Estimation of Value at Risk and ruin probability for diffusion processes with jumps

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We will consider a process X that satisfies

$$X_t = m + \int_0^t \sigma_s dB_s + \int_0^t b_s ds + \sum_{i=1}^{N_t} \gamma_{T_i^-} Y_i, \ t > 0, \tag{1}$$

$$X_t^* = \sup_{0 \le u \le t} X_u.$$

- We will obtain upper and lower bounds for

$$P[X_t^* > z]$$

- We will use these bounds for estimations to Ruin Pobabilities and VaR where

  For $q = 1 - \alpha$ in $]0, 1[$, the Value at Risk VaR associated with X_t^* will be given by

$$X_t = m + \int_0^t \sigma_s dB_s + \int_0^t b_s ds + \sum_{i=1}^{N_t} \gamma_{T_i^-} Y_i, \ t > 0,$$

where B is a one-dimensional Brownian Motion , N is a Poisson Process independent of B, $T_1, T_2, \dots,$ are the jump times for N, the random variables $Y_i, i \geq 1$ are i.i.d. and independent of the Poisson Process and the Brownian Motion.

We assume the following hypotheses and denote them by (**H**):

(1) b is an integrable process.

(2) For all $t > 0$, $E(\int_0^t \sigma_s^2 ds) < +\infty$.

(3) The jumps of the compound Poisson process are non-negative, i.e., $Y_1 \geq 0$ P-a.e., and we assume that Y_1 is not identically equal to 0.

(4) The process $(\sum_{i=1}^{N_t} \gamma_{T_i^-} Y_i, \ t > 0)$ is well-defined and integrable.

Hypothesis **UBï¿**$\frac{1}{2}$

(1) We assume that the law of the jumps admits a Laplace transform defined on $]-\infty, c[$ where c is a positive constant (or $c=+\infty$) and we put

$$L(x) = E[e^{xY_1}], \quad x < c.$$

(2) There exists $\gamma^* > 0$ such that $\gamma_s \leq \gamma^*$, P-a.s. for all $s \in [0,t]$.

(3) There exist $0 < \delta < (c/\gamma^*)$ and a constant $K_t(\delta) \geq 0$, such that, for all $s \in [0,t]$,

$$\delta \int_0^s b_u du + \frac{\delta^2}{2} \int_0^s \sigma_u^2 du + \lambda s (L(\delta\gamma^*) - 1) \leq K_t(\delta) \quad \text{a.e.} \qquad (2)$$

Let us remark that Assumption (**UB**)(3) is fulfilled if we replace it by the stronger one:

(3′) There exist constants $b^*(t) \geq 0$, $a^*(t) \geq 0$ such that,

$$\int_0^t \sigma_u^2 du \leq a^*(t), \int_0^s b_u du \leq b^*(t) \;\; P\text{-a.e. } \forall s \in [0,t].$$

In this case one has, for all $0 < \delta < c/\gamma^*$, The main result to get the upper estimate is:

Lemma

Assume (**UB**)*. Let* $\delta \in]0, c/\gamma^*[$ *be as in (3) of* (**UB**)*, and let* $z \geq m$*. Then,*

$$P(X_t^* \geq z) \leq \exp\{\delta(m - z) + K_t(\delta)\}.$$

Proof.

We have

$$\begin{aligned} P(X_t^* \geq z) &= P\left(\delta X_t^* \geq \delta z\right) \\ &\leq P\left(\sup_{0\leq s\leq t} M_s \geq \exp\{\delta(z-m) - K_t(\delta)\}\right), \end{aligned}$$

where $\{M_s,\ s \geq 0\}$ is the martingale defined by

$$M_s = \exp\left(\delta\left[\int_0^s \sigma_u dB_u + \sum_{i=1}^{N_t}\gamma^* Y_i\right] - \frac{\delta^2}{2}\int_0^s \sigma_u^2 du - \lambda s(L(\gamma^*\delta) - 1)\right).$$

$\{M_s,\ s \geq 0\}$ is a martingale since it is the product of a continuous martingale and a purely discontinuous one. The maximal inequality for exponential martingales gives the result.

Proposition

Let us assume (**UB**)*, and let*

$$A = \{\delta \in]0, c/\gamma^*[\ \mid \delta \textit{ satisfies } (\mathbf{UB})(3)\}.$$

Then

$$VaR_\alpha(X_t^*) \leq \inf_{\delta \in A} \left\{ m + \frac{K_t(\delta)}{\delta} - \frac{\ln \alpha}{\delta} \right\}.$$

Proof. Thanks to Lemma 1.1, $P(X_t^* > z) < \alpha$ is implied by

$$z > m + \frac{K_t(\delta)}{\delta} - \frac{\ln \alpha}{\delta},$$

which yields the result.

Hypothesis (**LB**):

There exist constants $b_*(t) \leq 0$, $a_*(t) > 0$, and $\gamma_* \in \mathbb{R}$, such that

$$\int_0^t \sigma_u^2 du \geq a_*(t)\ , \quad \int_0^s b_u du \geq b_*(t) \quad \text{and} \quad \gamma_s \geq \gamma_* \quad P\text{-a.e. } \forall s \in [0,t].$$

The lower bound depends on the sign of γ_*, so we shall discuss each case separately.

Lemma

Let us assume (**LB**), *and* $\gamma_* \leq 0$. *Then, for all* $z \in \mathbb{R}$ *and* $t \geq 0$,

$$P(X_t^* \geq z) \geq P(\sqrt{a_*(t)}|Z| + \gamma_* \sum_{i=1}^{N_t} Y_i \geq z - m - b_*(t)),$$

Sketch of the proof Proof

. Since by hypothesis the compound Poisson and the Brownian motion are independent, we use the product structure of the measure and adopt a natural notation, a change of time and the fact that we know the distribution of the sup of the Brownian Motion.

$$P(X_t^* \geq z) = \int_{\Omega^2} P^1(X_t^*(\cdot, w_2) \geq z) dP^2(w_2).$$

For fixed w_2, set $v = \sum_{i=1}^{N_t} Y_i(w_2)$. Then we have

$$P^1(X_t^*(\cdot, w_2) \geq z) \quad \geq \quad P^1(\sup_{s \in [0,t]} \int_0^s \sigma_u(\cdot, w_2) dB_u \geq z - m - b_*(t) - \gamma_* v).$$

As B is a P^1-Brownian motion,

$$R_s = \int_0^s \sigma_u(\cdot, w_2) dB_u$$

is a P^1-continuous martingale. By the change of time property, we know that there exists a P^1-Brownian motion $\tilde{B}$ such that

For $\gamma_* \geq 0$ we can give a lower bound which is always valid. For this, let us introduce the process

$$Y_s = m + \int_0^s b_u du + \int_0^s \sigma_u \, dB_u, \; s > 0.$$

As $\gamma_* \geq 0$, the process X dominates Y, so for all α, $VaR_\alpha(X_t^*) \geq VaR_\alpha(Y_t^*)$.

Proposition

If we assume hypotheses (**LB**), *and* $\gamma_* \geq 0$, *for all* α,

$$VaR_\alpha(X_t^*) \geq VaR_\alpha(Y_t^*) \geq m + b_*(t) + \sqrt{a_*(t)}\bar{\Phi}^{-1}(\alpha/2).$$

Now, if we make the additional hypothesis (denoted by (**LB**$'$)) that there exists a constant $a^*(t)$ such that

$$\int_0^t \sigma_u^2\, du \le a^*(t),$$

we can considerably improve the bounds.

Lemma

Under (**LB**), (**LB**$'$), *and* $\gamma_* \ge 0$, *for all* $z \in \mathbb{R}$ *and* $t \ge 0$,

$$\begin{aligned} P(X_t^* \ge z) &\ge P(\sqrt{a_*(t)}Z_1 - \sqrt{a^*(t) - a_*(t)}|Z_2| \\ &+ \gamma_* \sum_{i=1}^{N_t} Y_i \ge z - m - b_*(t)), \end{aligned}$$

where Z_1 *and* Z_2 *are* $N(0,1)$ *independent random variables that do not depend on* N *and* (Y_i).

We will assume in this section that the process $X_s,\ s \in [0,t]$ is given by

$$X_s = m + bs + \int_0^s \sigma_u dB_u - \sum_{i=1}^{N_s} \gamma_{T_i^-} Y_i, \quad m \geq 0, \tag{9}$$

where b is a constant and we assume that there exist constants a^* and $\gamma^* > 0$ such that

$$\forall s \in [0,t],\ \ \sigma_s^2 \leq a^*,\ \ \gamma_s \leq \gamma^* \ \text{ a.e.} \tag{10}$$

If $\gamma^* = 1$, we have the classical risk process (see [Asm, Gra]). If the insurer invests in a risky asset we obtain this general model, see for example [GaGrSc]. We can apply Lemma 1.1 to estimate the ruin probability.

Proposition

Let $\theta = E(Y_1)$. *Let* X *be as in (9), with coefficients bounded as in (10) above. Assume in addition that* $a^* > 0$ *and* $c = \infty$*; or that* $\lim_{x \to c/\gamma^*} L(x) = +\infty$*; and that the following safety load condition holds:*

$$b - \lambda\theta\gamma^* > 0. \tag{11}$$

Denote by δ^* *the greatest positive root in* $]0, c/\gamma^*[$ *of*

$$h(\delta) = -b\delta + \frac{\delta^2}{2}a^* + \lambda(L(\gamma^*\delta) - 1) = 0.$$

Then,

$$P(\sup_{0 \le s \le t} -X_s \ge 0) \le e^{-\delta^* m}.$$

As a consequence, we have the following upper bound for the ruin probability:

$$P(X_s \le 0 \text{ for some } s \text{ in }]0, \infty[) \le e^{-\delta^* m}.$$

The jumps have exponential law with parameter $\nu > 0$.

Corollary

Suppose (**UB**), (**LB**), *and* (**LB'**) *hold. Assume that for all* $s \in [0, t]$,

$$\gamma_* \leq \gamma_s \leq \gamma^*, \text{ a.e., with } \gamma^* > 0.$$

If $\gamma_* > 0$ *then*

$$\frac{\gamma_*}{\nu} \leq \liminf_{\alpha \to 0} \frac{VaR_\alpha(X_t^*)}{|\ln \alpha|} \leq \limsup_{\alpha \to 0} \frac{VaR_\alpha(X_t^*)}{|\ln \alpha|} \leq \frac{\gamma^*}{\nu}.$$

If $\gamma_* \leq 0$ *then*

$$\sqrt{2a_*(t)} \leq \liminf_{\alpha \to 0} \frac{VaR_\alpha(X_t^*)}{\sqrt{|\ln \alpha|}} \text{ and } \limsup_{\alpha \to 0} \frac{VaR_\alpha(X_t^*)}{|\ln \alpha|} \leq \frac{\gamma^*}{\nu}.$$

The geometric Brownian motion model.

Now we consider a case widely used in finance: imagine X satisfies

So for all $t \geq 0$,

$$Y_t = \ln m + \int_0^t \sigma_s dB_s + \int_0^t \tilde{b}_s ds + \sum_{i=1}^{N_t} \tilde{\gamma}_{T_i^-} Y_i,$$

where the processes $\tilde{b}$ and $\tilde{\gamma}$ are defined by

$$\tilde{b}_s = b_s - \frac{1}{2}\sigma_s^2,\ s \geq 0$$

and

$$\tilde{\gamma}_s = \sum_{i=0}^{+\infty} \frac{\ln(1+\gamma_s Y_i)}{Y_i} \mathbf{1}_{\{s \in]T_i, T_{i+1}]\}},$$

where $T_0 = 0$ and we adopt the convention that $\ln(1+0)/0 = 1$. If Y is integrable, we can apply all the results of the previous sections to get an estimate for $VaR_\alpha(Y_t^*)$ and hence for $VaR_\alpha(X_t^*)$, thanks to the following result, whose proof is now obvious:

Proposition

Let X be as in (12). Under (**GBM**), *for all* $\alpha \in]0,1[$, The work with heavy tails is in process and it will appear soon!!!!!!!!!!!

Thanks

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